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NUMERICAL STABILITY INVESTIGATION OF THE LAGRANGE SOLUTIONS OF AN ELLIPTIC RESTRICTED THREE-BODY PROBLEM

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We investigate numerically the triangular points of solutions of the elliptic restricted three-body problem. For the planar problem we have constructed, in the space of parameters e and μ (e is the eccentricity, μ is the ratio of the mass of the smaller of the two main bodies to the sum of their masses), the stability region for a majority of initial conditions and the region of formal stability. For resonant values of the parameters we found Liapunov-instability or stability in the fourth approximation relative to the coordinates and momenta of the perturbed motion. For spatial problems we have obtained a statement of stability in the fourth approximation.

1. We examine the motion of three material points attracted to each other by Newton's law. The differential equations of motion of the three-body problem allow a particular solution, corresponding to the motion under which the three bodies form an equilateral triangle rotating in their own plane around the center of mass of the three-body system. We investigate the stability of this particular solution for the case of the elliptic restricted three-body problem.

We consider the planar problem. We select the measurement units such that the distance between the bodies of finite mass, the sum of their masses, and the gravitational constant equal unity. Then in Nechvile coordinates with true anomaly ν as the independent variable, the expansion of the Hamiltonian function of the perturbed motion has the form

$$H = H_2 + H_3 + H_4 + \ldots \tag{1.1}$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 + \frac{1}{1 + e\cos\nu}\left(\frac{1}{8}q_1^2 - \frac{5}{8}q_2^2 - kq_1q_2\right) + \frac{e\cos\nu}{2(1 + e\cos\nu)}(q_1^2 + q_2^2)$$

$$H_3 = \frac{1}{1 + e\cos\nu}\left(-\frac{7\sqrt{3}k}{36}q_1^3 + \frac{3\sqrt{3}}{16}q_1^2q_2 + \frac{11\sqrt{3}k}{12}q_1q_2^2 + \frac{3\sqrt{3}}{16}q_2^3\right)$$

$$H_4 = \frac{1}{1+e\cos\nu}\left(\frac{37}{128}q_1^4 + \frac{25k}{24}q_1^3q_2 - \frac{123}{64}q_1^2q_2^2 - \frac{15k}{8}q_1q_2^3 - \frac{3}{128}q_2^4\right)$$

$$k = \frac{3\sqrt{3}}{4}(1-2\mu)$$

Here e is the orbit eccentricity of the main bodies, μ is the mass of the smaller one of them, and q_i, p_i are the generalized coordinates and momenta.

It is known [1, 2] that in the circular problem ($e = 0$) the Lagrange solutions are Liapunov-stable for all μ from the interval

$$0 < \mu < 1/18\,(9 - \sqrt{69}) \simeq 0.0385208\ldots$$

except for the two values $\mu_1 \simeq 0.02429\ldots$ and $\mu_2 \simeq 0.01352\ldots$ (third- and fourth-order resonances) for which instability takes place. For small values of eccentricity e it has been shown (*) that for nonresonant values of μ from the stated interval and for sufficiently small e the Lagrange solutions are stable for a majority of initial conditions, while for $0.024294\ldots < \mu < 0.0385208\ldots$ and for sufficiently small e, not belonging to the resonance curves of third and fourth order, the Lagrange solutions are formally stable. For values of parameters e and μ falling on third- and fourth-order resonances, we have obtained statements on stability in the fourth approximation and on Liapunov-instability.

At the present time it is not possible to solve analytically the problem of the stability of the Lagrange solutions for arbitrary values of the eccentricity. Even in the linear approximation the question can be answered only by numerical integration on modern electronic computers.

Danby [3] was the first to find numerically the region of stability in the first approximation in the plane of parameters e and μ. This region is shown in Fig. 1, where the third- and fourth-order resonant curves obtained in [4] are also presented. The resonant curves are numbered in accordance with the following values: *(1)* $4\lambda_2 = -1$; *(2)* $\lambda_1 + 3\lambda_2 = 0$; *(3)* $3\lambda_2 = -1$; *(4)* $\lambda_1 - 3\lambda_2 = 2$; *(5)* $2(\lambda_1 - \lambda_2) = 1$; *(6)* $\lambda_1 + 2\lambda_2 = 0$; *(7)* $3\lambda_1 + \lambda_2 = 2$; *(8)* $3\lambda_1 - \lambda_2 = 3$; *(9)* $2\lambda_1 + \lambda_2 = 1$; *(10)* $\lambda_1 + 3\lambda_2 = -1$; *(11)* $\lambda_1 - 2\lambda_2 = 2$; *(12)* $4\lambda_1 = 3$; *(13)* $3\lambda_2 = -2$.

Suppose that the characteristic indices $\pm i\lambda_1$ and $\pm i\lambda_2$ of a system with Hamiltonian H_2 are such that all its multiplicators are different. Then the system's Hamiltinian function H can be reduced to the form (see [5], for example)

$$H = \frac{1}{2}\lambda_1(q_1^2 + p_1^2) + \frac{1}{2}\lambda_2(q_2^2 + p_2^2) + \sum_{\nu_1+\nu_2+\nu_3+\nu_4=3}^{\infty} h_{\nu_1\nu_2\nu_3\nu_4}(\nu)\, q_1^{\nu_1}q_2^{\nu_2}p_1^{\nu_3}p_2^{\nu_4}$$

$$h_{\nu_1\nu_2\nu_3\nu_4}(\nu + 2\pi) = h_{\nu_1\nu_2\nu_3\nu_4}(\nu) \qquad (1.2)$$

by a real linear canonic transformation 2π-periodic in ν. In the present paper this transformation is found with the aid of the algorithm proposed in [6]. Further, if the condition

$$k_1\lambda_1 + k_2\lambda_2 \neq N \qquad (N \text{ is an integer}) \qquad (1.3)$$

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*) Markeev, A. P., Investigation of the stability of the Lagrange solutions of a planar elliptic three-body problem. Preprint IPM Akad. Nauk SSSR, №1, 1973.

is for integers k_1, k_2 satisfying the equality $|k_1| + |k_2| = k$, $k = 3, 4$, then there exists a canonic transformation, analytic in q_i, p_i and 2π-periodic in ν, reducing Hamiltonian (1.2) to the form

$$H = \lambda_1 r_1 + \lambda_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + H^* (q_i, p_i, \nu) \quad (1.4)$$
$$2r_i^2 = q_i^2 + p_i^2$$

Here the coefficients c_{20}, c_{11}, c_{02} do not depend on ν but only on the parameters e and μ, while the function H^*, 2π-periodic in ν, is analytic in q_i and p_i and its series expansion in the coordinates and momenta starts with terms of not less than fifth degree.

The reduction of the Hamiltonian function (1.2) to form (1.4) was effected not by the commonly-accepted Birkhoff transformation [5] but by the method of point mappings (*). The method of point mappings is based on the idea of the possibility of reducing the study of the motions of a dynamic system to the study of the properties of the generating functions prescribing the mapping. The reduction of Hamiltonian (1.2) to form (1.4) by the mapping method is accomplished in several stages. First, the mapping's generating function is found from the Hamiltonian function, next, it is normalized, i.e. reduced to some elementary form, and, finally, the normalized Hamiltonian function is constructed from the resulting normalized mapping. Such normalization method yields a significant economy to machine time.

Nevertheless, in order to find the coefficients c_{20}, c_{11}, c_{02} for every pair of parameters e and μ from the stability region of the linearized problem we have to integrate on the electronic computer systems of differential equations, at first, of 16th order (for finding λ_1 and λ_2 and the linear normalizing transformation), and afterwards, of 39th order (for obtaining the coefficients of the mapping's generating function). It turns out that for large values of eccentricity e and for small values of μ the computation time becomes very large for a prescribed accuracy. In this regard computations were not carried out for $e > 0.6$ and $\mu < 0.001$ in the region to the left of the value $\mu^c \simeq 0.028595...$ (the parametric resonance point $2\lambda_2 = -1$) and for $\mu > 0.042$ in the right region.

If $D \equiv c_{11}^2 - 4c_{20}c_{02} \neq 0$, the equilibrium position $q_i = p_i = 0$ is stable for the majority of initial conditions [7], while the motion in the neighborhood of the origin is conditionally periodic for these initial conditions.

In the present paper, besides Arnol'd stability we examine one more type of stability, namely, formal stability [8], signifying that instability cannot be detected by taking into account any finite number of terms in the expansion of the Hamiltonian function. For a system with Hamiltonian (1.4) formal stability, as was established by Glimm [9], holds only in the case of sign-definiteness of the quadratic form $F = c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2$ in the region $r_1 \geqslant 0$, $r_2 \geqslant 0$. As is easy to see, the form F is sign definite either if $D < 0$ or if $D > 0$ but all its coefficients c_{ij} have the same signs.

Figures 2 and 3 show the results of numerical calculations for parameters e and μ belonging to the stability region of the linearized problem. From these figures we see that Arnol'd stability holds in the whole region wherein the necessary stability conditions are satisfied, excepting the third- and fourth-order resonant curves (see Fig. 1) and the

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*) Markeev, A. P., On the point mapping method and some of its applications to the three-body problem. Preprint IPM Akad. Nauk SSSR, № 49, 1973.

curves $D = 0$, shown in dashed lines in Figs. 2 and 3. In the figures the regions of formal stability are hatched; the regions where $D < 0$ are depicted by a horizontal hatching, while the regions in which all coefficients c_{ij} have the same signs, by a sloped hatching (it turned out that there exist the only regions in which all the c_{ij} are positive).

2. We investigate the stability of the Lagrange solutions for third- and fourth-order resonances. First of all we note that the resonance curves $k_1\lambda_1 + k_2\lambda_2 = N$ in which k_1 and k_2 have different signs do not require a detailed investigation because for such resonances Moser [8] has proved the formal stability of the equilibrium position of a non-autonomous Hamiltonian system. Formal stability holds here when other resonances of any order are absent. In the contrary case we can make an assertion on the stability if we account for terms of up to only the fourth order, inclusive, in the expansion of the Hamiltonian function.

Let the problem's parameters e and μ be such that relation (1.3) is not satisfied when $|k_1| + |k_2| = 3$ i.e. a third-order resonance relation holds. In this case it is impossible to reduce the system's Hamiltonian function to form (1.4) because the presence of the resonance causes zero denominators to appear. The Hamiltonian function (1.2) can be reduced to the form [10]

$$H = a_{k_1k_2} r_1^{1/2|k_1|} r_2^{1/2|k_2|} \sin (k_1\varphi_1 + k_2\varphi_2) + H^* (r_i, \varphi_i, \nu)$$

where the polar coordinates r_i, φ_i are related to the coordinates q_i and to the momenta p_i by the relations

$$q_i = \sqrt{2r_i} \sin \varphi_i, \qquad p_i = \sqrt{2r_i} \cos \varphi_i \qquad (i = 1, 2)$$

the coefficient $a_{k_1k_2}$ does not depend on ν, while the function H^* is periodic in ν and its series expansion in powers of $\sqrt{r_i}$ starts with terms of not less than fourth degree. It has been shown [10] that if $a_{k_1k_2} \neq 0$, the equilibrium position $q_i = p_i = 0$ is unstable. If $a_{k_1k_2} = 0$, then third-order resonance does not appear in the third-degree terms of the series expansion of the Hamiltonian functions in powers of the coordinates and the momenta, and the stability investigation can be carried out as in Sect. 1.

In the stability region for the linearized restricted three-body problem, in the plane of the parameters e and μ there exist five resonant curves of third order, and on four of them the quantities k_1 and k_2 in the resonance relation (1.3) have the same signs. As a result of the numerical investigation carried out in the present paper it is clear that when $e > 0$ the coefficients $a_{k_1k_2}$ vanish nowhere for all four resonances $3\lambda_2 = -1$, $\lambda_1 + 2\lambda_2 = 0$, $2\lambda_1 + \lambda_2 = 1$, $3\lambda_2 = -2$, and, consequently, instability obtains on all the curves.

3. We now consider parameters e and μ for which relation (1.3) is not satisfied when $|k_1| + |k_2| = 4$. In this case the Hamiltonian function of the problem can be reduced to the form

$$H = c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + \tag{3.1}$$

$$b_{k_1k_2} r_1^{1/2|k_1|} r_2^{1/2|k_2|} \sin (k_1\varphi_1 + k_2\varphi_2) + H^* (r_i, \varphi_i, \nu)$$

Conditions for Liapunov-instability and for stability in the fourth approximation were obtained in [10] as functions of the coefficients $b_{k_1k_2}$ and c_{ij} of the Hamiltonian (3.1). Eight fourth-order resonant curves exist in the linearized problem's stability region. In six of them the k_1 and k_2 in the resonance relations $k_1\lambda_1 + k_2\lambda_2 = N$ have the same

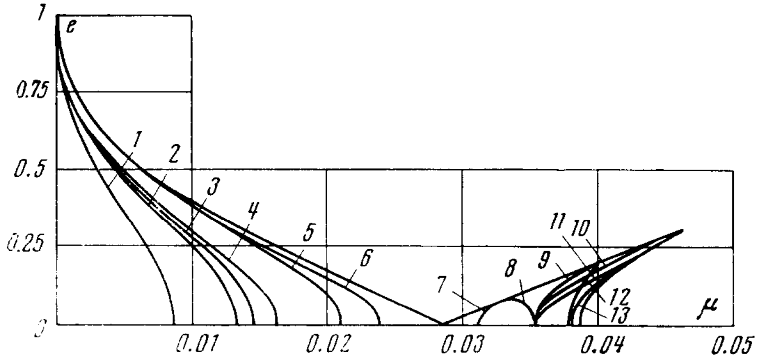

Fig. 1

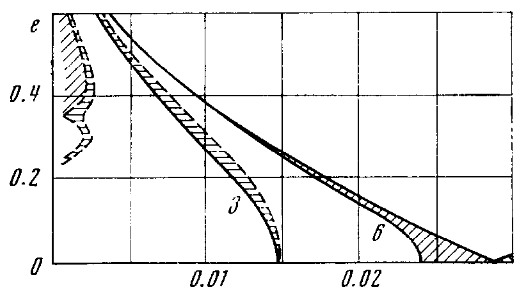

Fig. 2

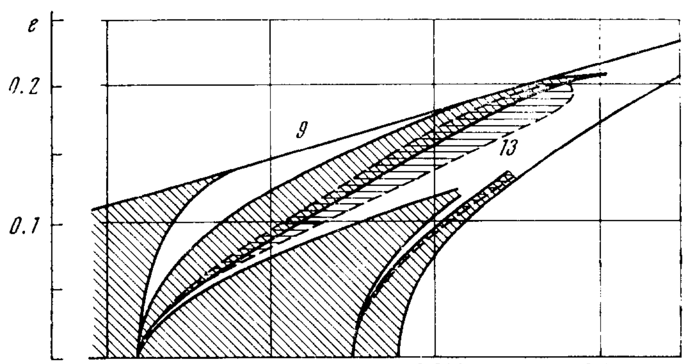

Fig. 3

Table 1

Number	Resonance	Interval of instability	Interval of stability in the fourth approximation
1	$4\lambda_2 = -1$	$0.022 < e < 0.611$	$0 < e < 0.022$ $0.611 < e < 0.8$
2	$\lambda_1 + 3\lambda_2 = 0$	$0 \leqslant e < 0.141$	$0.141 < e < 0.8$
5	$2(\lambda_1 + \lambda_2) = 1$	$0.026 < e < 0.45$	$0 < e < 0.026$
7	$3\lambda_1 + \lambda_2 = 2$		$0 < e < 0.065$
10	$\lambda_1 + 3\lambda_2 = -1$	$0.196 < e < 0.207$	$0 < e < 0.196$ $0.207 < e < 0.24$
12	$4\lambda_1 = 3$		$0 < e < 0.19$

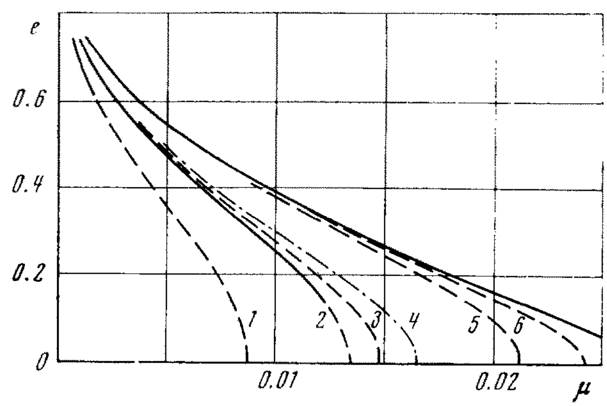

Fig. 4

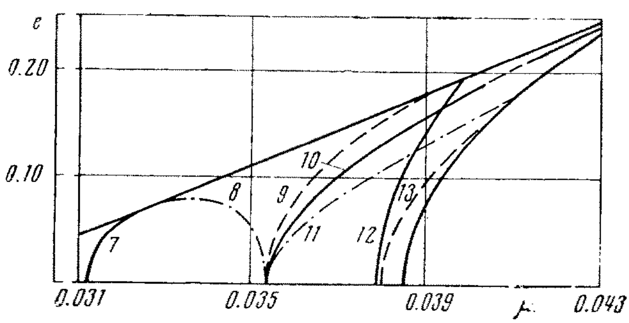

Fig. 5

signs. The calculations made show that both sections of stability in the fourth approximation as well as sections of Liapunov-instability exist on the fourth-order resonant curves. The results of the calculations are presented in Table 1.

All resonant curves of third and fourth order are shown in Figs. 4 and 5. On all curves, except $\lambda_1 + 2\lambda_2 = 0$ and $\lambda_1 + 3\lambda_2 = 0$, Liapunov-stability holds when $e = 0$ [2]. On the resonant curves $\lambda_1 - 3\lambda_2 = 2$, $3\lambda_1 - \lambda_2 = 3$, $\lambda_1 - 2\lambda_2 = 2$, shown by dashed-dotted lines on the figures, formal stability [8] holds when other resonance relations of any order are absent. On the third-order resonance curves $3\lambda_2 = -1$, $\lambda_1 + 2\lambda_2 = 0$, $2\lambda_1 + \lambda_2 = 1$, $3\lambda_2 = -2$ we stated Liapunov-instability when $e \neq 0$, as expected. These curves are depicted by dashed lines. On the fourth-order resonant curves $4\lambda_2 = -1$, $\lambda_1 + 3\lambda_2 = 0$, $2(\lambda_1 + \lambda_2) = 1$, $3\lambda_1 + \lambda_2 = 2$, $\lambda_1 + 3\lambda_2 = -1$, $4\lambda_1 = 3$ the sections of instability are shown by dashed lines while the sections of stability in the fourth approximation, by solid lines. The results obtained in the present paper coincide when $e = 0$ with the results in [1, 2], and with the results of the analytical investigation, mentioned earlier, when e is small.

4. Let us look at the spatial problem. Its peculiarity is that the spatial frequency of the linear oscillations equals unity for all e and μ and, consequently, always equals the frequency of the Kepler motion of the main bodies. It was proved in [11] that for small μ and e the presence of this resonance leads to the instability of the Lagrange sulutions. In the present paper we have clarified, as a result of the calculations made, that the Lagrange solutions are stable in the fourth approximation for all e and μ. Thus, we have not succeeded in finding the regions of instability. Apparently, if instability regions do exist, their boundaries pass very close to the third-order resonant curves or to the boundaries of the linear problem's stability region.

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CERTAIN STABILITY QUESTIONS IN THE PRESENCE OF RESONANCES

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We study questions of the stability of the equilibrium position of nonlinear systems neutral in the linear approximation. We obtain necessary and sufficient stability conditions in the presence of one resonance, as well as some results concerning the interaction of several resonances. We show that Liapunov instability follows from instability in finite order.

1. We consider a system of ordinary differential equations with real coefficients

$$dx_\alpha \, / \, dt = A_\alpha{}^\beta x_\beta + A_\alpha{}^{\beta\gamma} x_\beta x_\gamma + \ldots, \qquad \alpha = 1, \ldots, n \tag{1.1}$$

We study the stability of the equilibrium position $x_1 = \ldots = x_n = 0$ (relative to variations of the initial data) if the eigenvalues of the linearized system are purely imaginary, simple, and nonzero (Condition (A)) Under these conditions the question of the stability of the equilibrium position in the resonance-free case was examined by Molchanov (*). This question has been studied for Hamiltonian systems in the presence of resonances of arbitrary order [1]. The case of one third-order resonance was considered in [2] for general systems. In the present paper we have obtained necessary and sufficient conditions for the stability of the equilibrium position of system (1.1) in second order by perturbation theory in the presence of parametric resonance . We have proved the Liapunov-instability of the equilibrium position of system (1.1) in the presence of an arbitrary third-order resonance if the system is Birkhoff-unstable (in second order) and we have examined the question of the interaction of two or of several resonances. In particular, we have shown that the interaction of two resonances can lead to instability even when each resonance individually does not cause instability.

Let $\lambda_1, \ldots, \lambda_l, \ -\lambda_1, \ldots, -\lambda_l$ be the eigenvalues (frequencies) of the system being analyzed $(2l = n)$. We say that system (1.1) possesses kth-order resonance if integers k_m $(m = 1, \ldots, l)$, exist, not all equal to zero, $|k_1| + \ldots + |k_l| = k$, such that $k_1\lambda_1 + \ldots + k_l\lambda_l = 0$. (For example, relations of the form

$$\lambda_i - 2\lambda_j = 0, \ \lambda_i + \lambda_j + \lambda_k = 0, \ \lambda_i + \lambda_j - \lambda_k = 0$$

exhaust all third-order resonances). The vector $(k_1, \ldots, k_l)$ is said to be resonant.

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*) Molchanov, A. M., On the stability of nonlinear systems. Thesis for a Doctor's degree, Moscow, 1962.